

Analysis about efficiency of a circular linear process

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Abstract

Cycle efficiency is an important part of teaching of thermodynamics. However, introduction of this topic is often limited to cycles consisting of standard processes only, as discussed in most textbooks. Here, we calculated the efficiency of a circular linear process, which is an easy but clear example to illustrate the approaches to non-standard processes. We found that the key to a simple and didactic solution is applying differential form of the first law of thermodynamics for determination of conversion point of endothermic and exothermic processes. In addition, we obtained the results with different working media, i.e., ideal gas and Van der Waals gas, and compared their characteristics.

Keywords: Thermodynamics, circular linear process, endothermic and exothermic characteristics, cycle efficiency

1 Introduction

In the learning of college thermodynamics courses, students tend to hold robust confusion about the key conceptions such as heat, temperature, internal energy, work and the first law of thermodynamics which reveals the relations among them (Dreyfus et al., 2015; Sözbilir, 2003; Goldring & Osborne, 1994; Loverude et al., 2002; Miller et al., 2005; Meltzer, 2007; Kautz & Schmitz, 2007; Leinonen et al., 2009). Many textbooks introduce how the knowledge can be applied in practice by discussing circular process based on heat engines and further extending it to introduce cycle efficiency, which is of crucial importance in both science and engineering. Most of them limits their discussion to Carnot engine with ideal gas as the working medium in the main text (Feynman et al., 1963), while those dealing with other types of heat engines (e.g. Otto engine) are covered in supplementary exercise part, if any (Reif, 2010; Huang, 2009). However, the respective circles only consist of standard processes. As a consequence, it is easy to distinguish between endothermic and exothermic parts of the circle.

In this article, we first discuss a circular process consisting of an isochoric process, an isobaric process and a linear process with negative slope in $p - V$ curve (see $X \rightarrow Y \rightarrow Z \rightarrow X$ in Fig. 1, circular linear process afterwards), and calculate the respective efficiency. It is highlighted that this seemingly simple-to-analyze circular process is in fact error-prone. In obtaining the solution, students need to learn how to distinguish between endothermic and exothermic parts of the linear process, which helps them build correct understanding of endotherm and exotherm. The key to distinction we provide is applying the differential form of the first law, which is both easy to handle in mathematics and easy to understand in physics. In addition, calculating cycle efficiency *per se* requires students to hold deep insight of its definition, that is, the ability to distinguish among the gross absorbed heat, the gross released heat and the

net heat. After we conclude our calculation for ideal gas, we apply the same method to Van Der Waals gas, and discuss the similarity and difference between them.

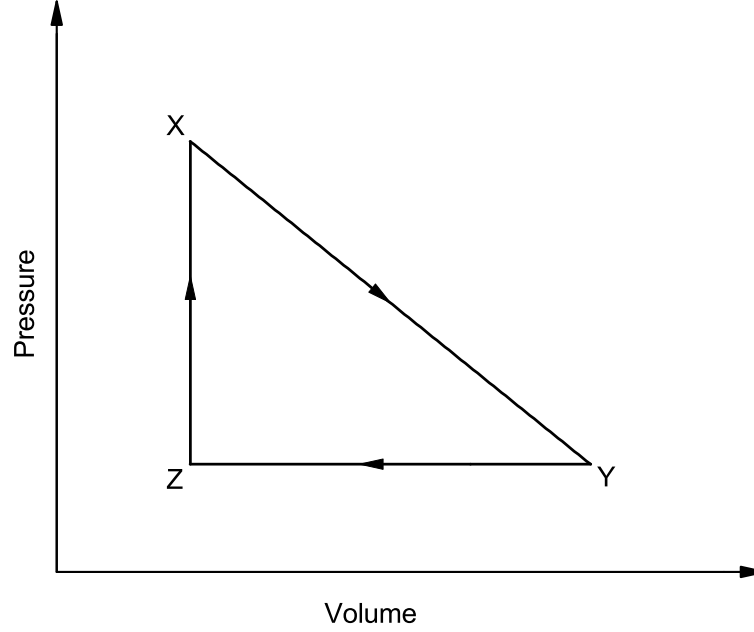


Figure 1. The $p - V$ curve of a circular linear process

2 Circular linear process with ideal gas

We first discuss the circular linear process with working medium as ideal gas.

According to the definition of efficiency of heat engine

$$\eta = 1 - \frac{Q_1}{Q_2} \quad (1)$$

we need to calculate gross absorbed heat Q_1 and gross released heat Q_2 respectively. As to the cycle shown in Fig. 1, gas releases heat in the isobaric process and absorbs heat in the isochoric process. However, it is not that easy to analyze the endothermic and exothermic characteristics of the linear part since there exists a conversion point in it. That is, the linear process consists of both endothermic part and exothermic part divided by a point on it. A common fallacy is assuming the whole process as endothermic or exothermic and calculating the net heat instead of absorbed heat and released heat separately, which leads to wrong results.

Here, we first determine where the conversion point exists using the differential form of the first law to make the endothermic part and exothermic part clear. Denote i as the degree of freedom of ideal gas and ν as the amount of substance. And let the linear equation in the circular linear process be

$$p = KV + B \quad (2)$$

State equation of ideal gas

$$pV = \nu RT \quad (3)$$

Then, the increment of internal energy of 1mol ideal gas is

$$dE = \frac{i}{2} (pdV + Vdp) \quad (4)$$

The differential form of the first law

$$\delta Q = dE + pdV \quad (5)$$

If we let $\nu = 1\text{mol}$, then according to Eqs. (2) to (5), we can get

$$\delta Q = \left[(1+i)KV + \left(1 + \frac{i}{2}\right)B \right] dV \quad (6)$$

Let $\delta Q = 0$, we can determine the volume and pressure of the conversion point. The only root to the equation is

$$V_g = -\frac{i+2}{i+1} \frac{B}{2K}, p_g = \frac{i}{i+1} \frac{B}{2} \quad (7)$$

which shows that there exists only one conversion point.

As is shown in figure 2, the linear part of circular linear process is tangent with an adiabat on G.

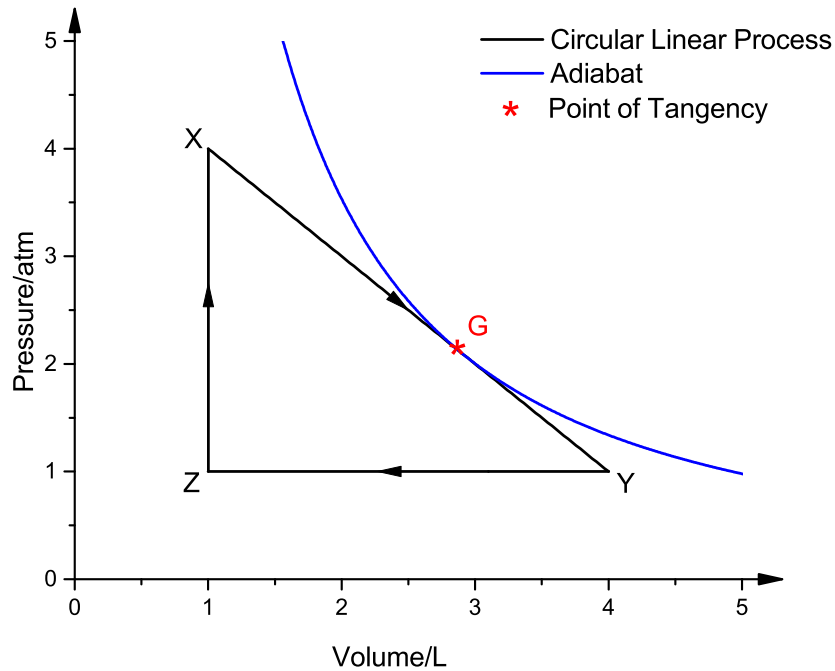


Figure 2. The linear part of circular linear process (ideal gas) tangent with an adiabat

In Fig. 2, denote $X(p_x, V_x)$, $Y(p_y, V_y)$ and $Z(p_z, V_z)$. It is obvious that $V_x = V_z$, $p_y = p_z$. And we can express the slope of the linear part as

$$K = \frac{p_y - p_x}{V_y - V_x} \quad (8)$$

We calculate the absorbed heat and released heat separately for the three parts.

1. Isobaric process $Y \rightarrow Z$: Gas releases heat

$$Q_{Y \rightarrow Z} = \Delta E_{Y \rightarrow Z} + W_{Y \rightarrow Z} = \frac{i+2}{2} p_y (V_z - V_y) \quad (9)$$

2. Isochoric process $Z \rightarrow X$: Gas absorbs heat

$$Q_{Z \rightarrow X} = \Delta E_{Z \rightarrow X} + W_{Z \rightarrow X} = \frac{i}{2} V_x (p_x - p_z) \quad (10)$$

3. Linear process: The volume and pressure of conversion point are showed in Eq. (7). When the state of the ideal gas changes from X to G along the straight line, the gas absorbs heat. The quantity of the absorbed heat is

$$Q_{X \rightarrow G} = \frac{i}{2} (p_g V_g - p_x V_x) + \int_{V_x}^{V_g} p dV \quad (11)$$

When the state of the ideal gas changes from G to X along the straight line, the gas releases heat. The quantity of the released heat is

$$Q_{G \rightarrow Y} = \frac{i}{2} (p_y V_y - p_g V_g) + \int_{V_g}^{V_y} p dV \quad (12)$$

Thus, the gross released heat is

$$Q_1 = \frac{i+1}{2} p_g V_g - \frac{1}{2} p_x V_x - \frac{1}{2} p_g V_x + \frac{1}{2} p_x V_g - \frac{i}{2} p_y V_x \quad (13)$$

The gross absorbed heat is

$$Q_2 = \frac{i+2}{2} p_y V_x - \frac{1}{2} p_y V_y - \frac{i+1}{2} p_g V_g - \frac{1}{2} p_y V_g + \frac{1}{2} p_g V_y \quad (14)$$

The efficiency of the circular linear process is

$$\eta = \frac{(p_y - p_x)^2}{iK (p_y V_x - p_g V_g) + (p_x^2 - p_g^2)} \quad (15)$$

If we let the values of volume and pressure of points X , Y and Z as depicted in Fig. 2, that is $p_x = 4atm$, $V_x = 1L$, $p_y = 1atm$, $V_y = 4L$, $p_z = 1atm$, $V_z = 1L$, then from Eq. (15), the efficiency is

$$\eta \approx 24.29\% \quad (16)$$

3 Circular linear process with Van Der Waals gas

When dealing with some more practical problems, differences between ideal gas and actual gas become significant, making it impossible to approximate actual gas with ideal gas model. That's why Van Der Waals introduced a more accurate model to make a better approximation. With Van Der Waals gas as working medium, textbook thermodynamics can be linked more closely with practical situations.

For convenience of comparison, we change the working medium to Van Der Waals gas while still discuss the same cycle with identical parameters.

State equation of Van Der Waals gas

$$p = \frac{RT}{V-b} - \frac{a}{V^2} \quad (17)$$

The increment of internal energy of $1mol$ Van Der Waals gas is

$$dE = C_v dT + \frac{a}{V^2} dV \quad (18)$$

Applying the differential form of the first law, we have

$$\delta Q = dE + p dV \quad (19)$$

From Eqs. (17) to (19), it follows that

$$\delta Q = \left\{ \left(\frac{2C_v K}{R} + K \right) V + \left(\frac{C_v B - C_v b K}{R} + B \right) + \left(1 - \frac{C_v}{R} \right) \frac{a}{V^2} + \left(\frac{2C_v a b}{R} \right) \frac{1}{V^3} \right\} dV \quad (20)$$

Let $\delta Q = 0$, we can obtain conversion point(s) of the endothermic part(s) and the exothermic part(s) of the linear process as we already do for ideal gas. If we suppose the endothermic characteristics of Van Der Waals gas is consistent with ideal gas, that is, only one conversion point with physical practice exists on the line, then similar to calculation of the ideal gas efficiency, we can get

1. Isobaric process $Y \rightarrow Z$: Gas releases heat

$$Q_{Y \rightarrow Z} = \Delta E_{Y \rightarrow Z} + W_{Y \rightarrow Z} = \left(C_v T_y - \frac{a}{V_y} \right) - \left(C_v T_z - \frac{a}{V_z} \right) + (V_y - V_x) p_y \quad (21)$$

2. Isochoric process $Z \rightarrow X$: Gas absorbs heat

$$Q_{Z \rightarrow X} = \Delta E_{Z \rightarrow X} = \left(C_v T_x - \frac{a}{V_x} \right) - \left(C_v T_z - \frac{a}{V_z} \right) \quad (22)$$

3. Linear process:

$X \rightarrow G$: Gas absorbs heat

$$Q_{X \rightarrow G} = \Delta E_{X \rightarrow G} + W_{X \rightarrow G} = \left(C_v T_g - \frac{a}{V_g} \right) - \left(C_v T_x - \frac{a}{V_x} \right) + \frac{1}{2} (V_g - V_x) (p_g + p_x) \quad (23)$$

$G \rightarrow Y$: Gas releases heat

$$Q_{G \rightarrow Y} = \Delta E_{G \rightarrow Y} + W_{G \rightarrow Y} = \left(C_v T_g - \frac{a}{V_g} \right) - \left(C_v T_y - \frac{a}{V_y} \right) - \frac{1}{2} (V_y - V_g) (p_g + p_y) \quad (24)$$

From what we have discussed, the gross released heat can be expressed as follows

$$Q_1 = Q_{G \rightarrow Y} + Q_{Y \rightarrow Z} \quad (25)$$

The gross absorbed heat

$$Q_2 = Q_{X \rightarrow G} + Q_{Z \rightarrow X} \quad (26)$$

Net heat of the linear process

$$Q' = |Q_{X \rightarrow G} - Q_{G \rightarrow Y}| \quad (27)$$

Again, it is error-prone here as students tend to add the net heat Q' to the gross absorbed heat or the gross released heat. However, ever since we note the fact that there are the endothermic part and the exothermic part of linear process which is divided by the conversion point, the efficiency is clearly expressed as

$$\eta = \frac{(p_y - p_x)^2}{iK(p_y V_x - p_g V_g) + (p_x^2 - p_g^2) - iKab(V_x^{-2} - V_g^{-2}) - Ka(i-2)(V_g^{-1} - V_x^{-1}) - iKb(p_y - p_g)} \quad (28)$$

It is obvious that when $a = b = 0$, the efficiency obtained from Eq. (28) is identical to the ideal gas version Eq. (15).

Now we use the same parameter values for Van Der Waals gas, i.e., $p_x = 4atm$, $V_x = 1L$, $p_y = 1atm$, $V_y = 4L$, $p_z = 1atm$, $V_z = 1L$, $K = -1$, $B = 5$, and let $a = 3.6atm \cdot L^2$, $b = 0.043L$. The numerical solutions of Eq. (20) are

$$V_{g1} = 2.8489L, V_{g2} = 0.5284L, V_{g3} = 0.1409L, V_{g4} = -0.5669L \quad (29)$$

As is obvious in Fig. 3, the linear process is tangent with the adiabat family on three points. Thus, it is tempting to reach the conclusion that there are three conversion points for the Van Der Waals gas instead of one for the same cycle. However, we need to realize that when actual gas compresses, gas phase transition occurs, which leads to an unknown amount of reduction of the mass of gas, invalidating all equations above. The respective process is depicted as concave part of adiabat 2 in Fig. 3, which is tangent with the extension part of linear process on G_3 . Therefore, it is meaningless to include G_3 in the linear process. So it is with G_2 , as even though it is not on the concave part, the distance between them is too short to determine whether the working medium is still totally gaseous, which further reduces the number of practical conversion points to one, that is G_1 . In other words, there is only one conversion point of practical significance, which corresponds to our hypothesis before and gives us a better understanding of the association between the two models.

Finally, let V_g in Eq. (28) equal to V_{g1} in Eq. (29), we have

$$\eta \approx 29.57\% \quad (30)$$

Comparing the calculation procedure of ideal gas with the one of van der Waals gas, we can find that although there are differences between the ideal gas and van der Waals gas model, when applied to the same linear process, the two models still have much in common, such as same number of practical conversion points. From the extending discussion of Van Der Waals gas, we hope students can form a unified and systematic understanding of gas instead of only limiting their eyesight to description of ideal gas or isolating their ideas of different gas models, as well as realize the differences between abstract physical objects and practical situations.

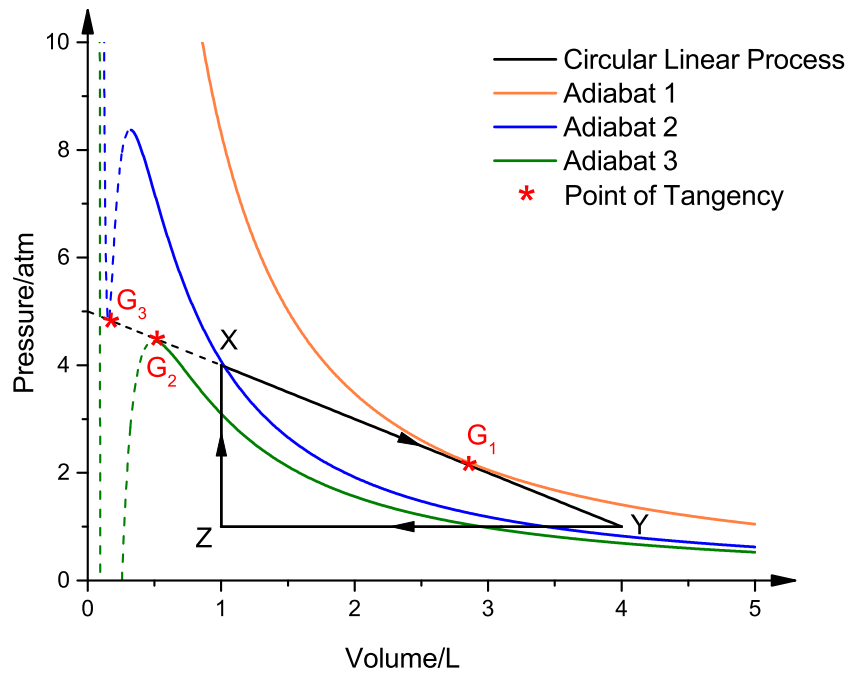


Figure 3. The linear part of circular linear process (Van Der Waals gas) tangent with adiabats

4 Conclusion

This article provides a didactic example in teaching thermodynamics: the calculation of efficiency for a cycle consisting of an isochoric process, an isothermal process and a linear process with negative slope in $p - V$ curve. In the procedure, conceptions such as gross absorbed heat, the gross released heat and the net heat are well demonstrated, which helps enable students to distinguish between endothermic and exothermic processes. In addition, students are expected to gain a better understanding of the insight of first law of thermodynamics by learning how to apply its differential form to solve problems. As to working medium, the models for ideal gas and Van Der Waals are discussed separately. However, post-hoc comparison shows that although mathematically endothermic and exothermic characteristics differ enormously, as while for ideal gas there is only one endothermic conversion point, multiple points exist for the Van Der Waals counterpart, in physics they're not practical. In fact, only one conversion point is possible whatever the working medium, since they are both actually an approach to real gas, the difference between which only lies in accuracy and scope of application. This comparison provides opportunities for students to gain some insight of the relationship between ideal gas and real gas.

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