For office use only	Team Control Number	For office use only
T1 ________	**007**	F1 ________
T2 ________		F2 ________
T3 ________	Problem Chosen	F3 ________
T4 ________	**A**	F4 ________

2017 Mathematical Contest in Modeling (MCM) Summary Sheet
(Attach a copy of this page to your solution paper.)

Bath as the Roman Bath: Optimazing Strategies for Reheating Bathtub Based on Water Thermodynamics

Summary

Private bath is readily available thanks to invention of modern bathtubs. However, taking a comfortable hot bath economically still requires some tricks.

In this report, we provide a series of strategies that optimize water usage while realizing satisfactory experience for bathers. Specifically, we accomplish the following:

- **Water Thermodynamics:** By applying Fourier's Law of heat transfer and Newton's Law of cooling, we establish the thermodynamics model of bathtub water. Temperature distribution over space and time is determined numerically from it through finite difference methods.
- **Temperature Requirements:** Apart from average temperature in the bathtub, we also provide a measure for temperature difference. The temperature requirements are a combination of these factors and are tailored specificly to different conditions.
- **Criteria for Strategies:** Based on four practical criteria, we rule out most of impractical strategies and show special consideration for convenience of bathers. The optimal strategies are both easy to handle and economical.
- **Versatility:** We explore variations of bathtub, bather and additive scenarios that realistically simulate real-life situaions. This adaptability is achieved through the incorporation of adjustable parameters and individually dependent factors.

Ultimately, our model demonstrates an easy-to-handle strategy that faithfully respect bather's relaxational and sanatational requirements and consumes minimum amount of water.

Contents

1 Introduction

1.1 Overview

No matter it's for sanitation or recreation purpose, hot bath enjoys its wide popularity around the globe, wherever it is possible. Back in ancient Rome, most cities had at least one, if not many, *thermae* (means public bath complexes in Greek) accessible to its citizens [1]. Nowadays, bathtubs are readily available for private bathing.

Figure 1: The Roman Baths in Bath, England [2]. Buildings above street level were constructed over ancient Roman *thermae* in 19th century.

Figure 2: Modern private bathtub [3]. Facuet and overflow drain are in the left end of the bathtub.

To keep the bath warm, either a secondary heating system must be present or hot water must be added once in a while. In the latter case, when and how much hot water is added to the bathtub is of importance. Insufficient water supply leads to uncomfortable experience, while excessive amount of water goes wasted. In addition, the temperature of newly added water should be chosed wisely. Since hot water is often added from a faucet in the end of the bathtub, it takes time for the temperature turns even in the bath through heat transfer. Pouring extreme hot water generally saves total amount of water used, but large temperature difference impedes relaxation and can even cause burns.

In this report, our primary objective is to find the best strategy of adding hot water to a bathtub such that minimum amount of water is wasted through overflow drain while both the average temperature and temperature difference are controlled in an acceptable range. To accomplish this, we first establish a general model of temperature of bathtub under an ideal condition, with a cuboid bathtub shape, a cuboid bather shape and no bather motions invovled. A series of greedy-based practical strategies are then formulated. Different bathtub shapes and bather postures, as well as bath addictive are further discussed to accomodate to a real-life situation. For each specific condition, its optimal strategy is highlighted.

1.2 Nomenclature

The nomenclature used in this report is listed by section as follows:

Symbol	**Definition**
Section 2.1	
L, H, W	Length, width and height of bathtub.
l, h, w	Length, width and height of bather when lying flat.
x_b, y_b, z_b	Cartesian coordinates of the center of bather.
x_e, y_e, z_e	Cartesian coordinates of the entering position of hot water.
V	Space in the bathtub filled with water.
C	Specfic heat capacity of water.
λ	Coefficient of heat conduction of water.
ρ	Density of water.
k_a	Convective heat tranfer coefficient between water and air.
k_b	Convective heat tranfer coefficient between water and bathtub.
T_e	Environment temperature.
T_f	Floor temperature.
T_h	Hot water temperature.
$T(x, y, z, t)$	Temperature of water at (x, y, z) and time t.
Δm	Mass of hot water poured into the bathtub per unit time.
Section 2.2	

$\overline{T}(t)$	Average temperature in the bathtub at time t.
$\Delta T(t)$	Standard deviation of temperature over the bathtub at time t.
T_{min}, T_{max}	acceptable temperature range of hot water immersion.
α	Individually dependent factor measuring bather's tolerance of temperature difference.
Δm_{max}	Maximum water flow through faucet.
Δm_i	Δm in ith period.
N	Total number of periods of adding hot water in a single bath.
t_0	Total bathing time.
Δt_i	Length of ith period.

Section 2.3

a, b, H'	Semi-major axis, semi-minor axis and height of the oval cylinder bathtub.
θ	Parameter of oval equation.
l_1, l_2	Geometric parameters of the sitting bather.
k_f	Convective heat transfer coefficient between water and foam.

1.3 Assumptions

The reliability of our models depends on certain key, simplifying assumptions, which are listed below:

- **Constant and even environment and floor temperature.** Since certain amount of hot water is added to bathtub prior to bathing, we assume that the dramatic change of environment and floor temperature happens in that period only and remains constant afterwards due to thermal equilibrium. In addition, the temperature is considered even for every surface of the bathtub, equal to the environment temperature in the flank and floor temperature in the bottom.
- **Evaporation equilibrium.** Similar to previous assumption, it is assumed that evaporation equilibrium has been reached before bathing due to pre-added water.
- **Insulated bather's body.** Although actually, bather is absorbing heat from water when bathing, its effect is considered insignificant as the temperature difference between body and water is trivial compared to that between water and enviroment. Therefore, the heat transfer between body and water is neglected.
- **Always full tub.** The initial amount of water in the bathtub is a personal preference and the repsective strategy before the tub is full is individually independent. We thus focus on the general strategy when the bathtub is already filled up.

- **Incompressible water.** Water is assumed to be incompressible. Under ordinary conditions, its compressibility ranges from 4.4 to $5.1 \times 10^{-10} Pa^{-1}$ [4], which is neglectable.
- **No radiation.** The radiation of water as well as other materials invovled are trivial compared to conduction and convection with respect to heat transfer.
- **Constant water flow.** It is impractical to adjust the faucet incessantly, either by hand or a computer-based control system. We assume that once the facuet is open, the water flow stays constant for a period until it is changed again.
- **Added water viewed as point heat flux source.** The dynamics after the newly added water from the faucet is complicated and dependent on several factors including the height and shape of the faucet, the initial velocity of the water, etc. We circumvent these by viewing the added hot water as point heat flux source at the position where hot water enters the bathtub. Its power is assumed equal to the released heat of the added water from its origin temperature in faucet to present temperature of the entering position in unit time.
- **Stationary foam layer.** When bubble additives are added to bathtub, the foam layer is assumed to remain stationary on the top of the bathtub, and the loss of bubbles are neglected.

2 Model Theory

2.1 Water Thermodynamics

We first consider an over-simplified version of the problem. Both the bathtub and the bather are of cuboid shapes, with geometric parameters L, W, H for bathtub and l, w, h for bather. In 3-D Cartesian coordinate system, the center of the bathtub is set at $(W/2, L/2, H/2)$ and the center of bather is set at (x_b, y_b, z_b). The bather lies flat *in* the water, parallel to its bottom. This requires

$$x_b \in [-\frac{W}{2} + \frac{w}{2}, \frac{W}{2} - \frac{w}{2}] \tag{1}$$

$$y_b \in [-\frac{L}{2} + \frac{l}{2}, \frac{L}{2} - \frac{l}{2}] \tag{2}$$

$$z_b \in [-\frac{H}{2} + \frac{h}{2}, \frac{H}{2} - \frac{h}{2}] \tag{3}$$

The hot water enters the bathtub at (x_e, y_e, z_e), which, according to our point heat flux source assumption, is the position of the heat source. Note that the z_e is not exactly H considering the entering velocity at the surface. Normally, z_e should be slightly less than H. The above geometric setting is illustrated in Figure 3.

The thermodynamics invovled in this simplfied problem includes

- heat transfer in bathtub water,
- heat transfer between water and bather,

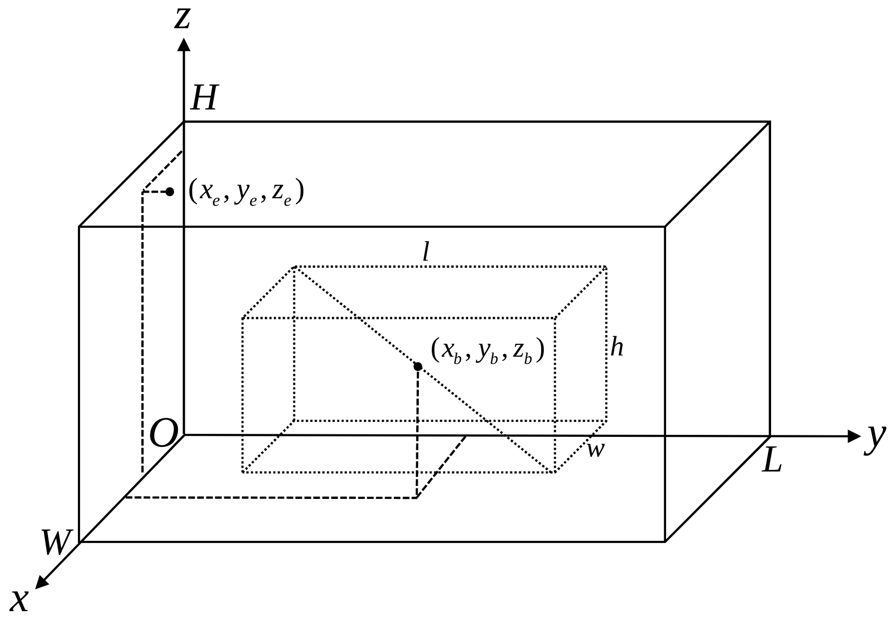

Figure 3: Geometric setting of bathtub, bather and heat source.

- heat released to enviroment through bottom and four side surfaces via convection and through the top surface via convection and evaporation,
- and heat absorbed from added hot water.

The details are explained in following subsections.

2.1.1 Inner Heat Transfer

General temperature model inside fluids invovles both conductive heat transfer and convective heat transfer. The conductive heat transfer is well modeled by Fourier's Law, while the convective heat transfer is much more difficult to analyze and an index, Rayleigh number is introduced to indicate the onset of convection [5]. In general, fluids at the boundary layer is dominated by convection, while the inner bulk remain approximately stationary and convective heat is neglectable compared to conduction heat. Therefore, for the inner water thermodynamics, we only consider heat conduction and, considering water is assumed to be incompressible, we apply Fourier's Law,

$$\rho C \frac{\partial T}{\partial t} = \lambda(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}),$$

$$(x, y, z) \in V = [0, W] \times [0, L] \times [0, H] - [x_b - \frac{w}{2}, x_b + \frac{w}{2}] \times [y_b - \frac{l}{2}, y_b + \frac{l}{2}] \times [z_b - \frac{h}{2}, z_b + \frac{h}{2}] \quad (4)$$

where $T(x, y, z, t)$ is the temperature at (x, y, z) and time t, C is the specific heat capacity of water, ρ is the density of water and λ is the coefficient of heat conduction of water. The range of (x, y, z), V is illustrated in Figure 3, the space in the outer cuboid excluding the inner cuboid.

2.1.2 Heat Transfer between Water and bather

As stated in the assumption part, the bather's body is considered approximately insulated from water. Therefore, the boundary conditions at the bather's body are

$$\frac{\partial T}{\partial x} = 0, x = x_b \pm \frac{w}{2}, y_b - \frac{l}{2} < y < y_b + \frac{l}{2}, z_b - \frac{h}{2} < z < z_b + \frac{h}{2} \tag{5}$$

$$\frac{\partial T}{\partial y} = 0, x_b - \frac{w}{2} < x < x_b + \frac{w}{2}, y = y_b \pm \frac{l}{2}, z_b - \frac{h}{2} < z < z_b + \frac{h}{2} \tag{6}$$

$$\frac{\partial T}{\partial z} = 0, x_b - \frac{w}{2} < x < x_b + \frac{w}{2}, y_b - \frac{l}{2} < y < y_b + \frac{l}{2}, z = z_b \pm \frac{h}{2} \tag{7}$$

which, as illustrated in Figure 4, indicates the normal heat flux through all six faces of the inner cuboid are zero.

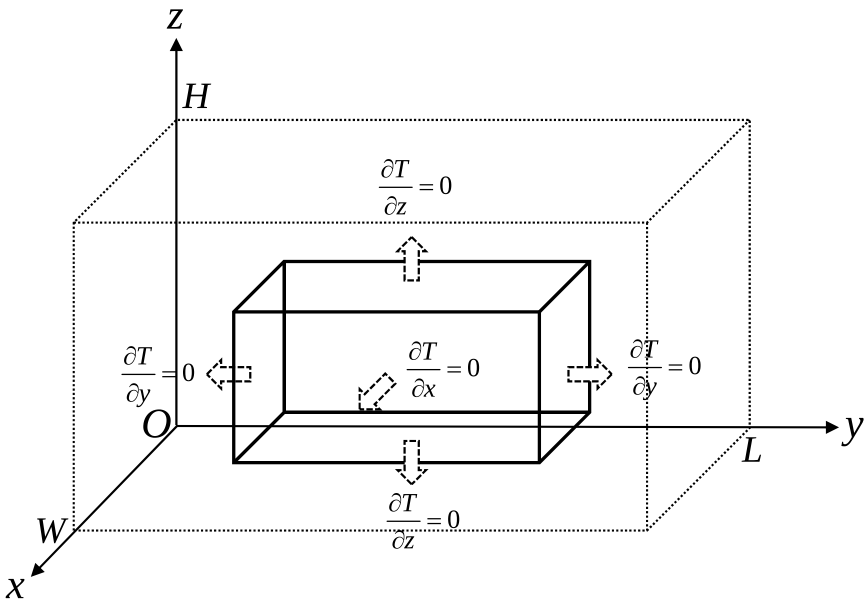

Figure 4: Heat transfer between water and bather.

2.1.3 Released Heat through Sides

The heat released through flank and bottom surfaces is modeled by convective heat transfer model, and regulated by Newton's law of cooling [6], which states the rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings. In our case, denote k_b as the convective heat transfer coefficient between water and bathtub, T_e as the constant environment temperature, T_f as the constant floor temperature. The boundary conditions in the flanks band bottom are expressed as

$$\lambda \frac{\partial T}{\partial x} = k_b(T - T_e), x = 0, 0 < y < L, 0 < z < H \tag{8}$$

$$-\lambda \frac{\partial T}{\partial x} = k_b(T - T_e), x = W, 0 < y < L, 0 < z < H \tag{9}$$

$$\lambda \frac{\partial T}{\partial y} = k_b(T - T_e), 0 < x < W, y = 0, 0 < z < H \tag{10}$$

$$-\lambda\frac{\partial T}{\partial y} = k_b(T - T_e), 0 < x < W, y = L, 0 < z < H \tag{11}$$

$$\lambda\frac{\partial T}{\partial z} = k_b(T - T_f), 0 < x < W, 0 < y < L, z = 0 \tag{12}$$

As illustarted in Figure 5, conditions above describe the relationships between the normal heat flux in flank and bottom sides and respective temperature outside.

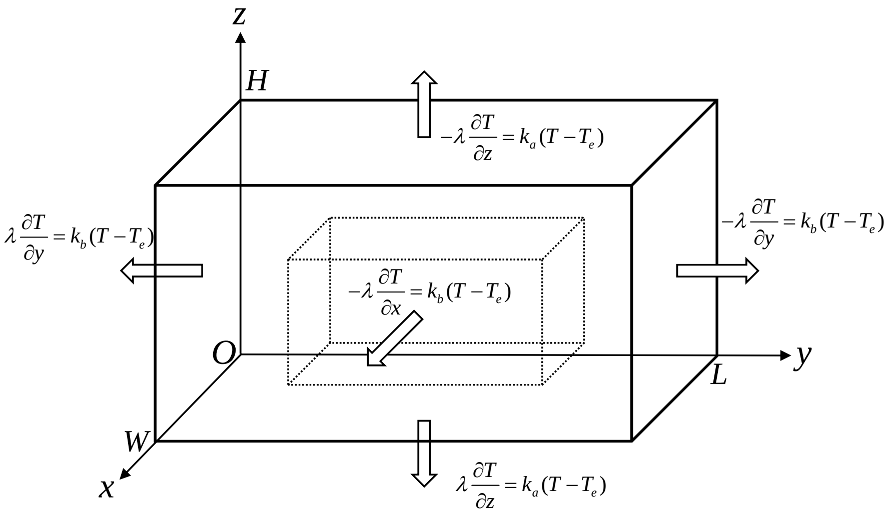

Figure 5: Released heat through sides.

The heat released in the top is generally difficult to calculate as it invovles evaporation process. However, as we assume before, the evaporation process already reaches equilibrium state before bathing. Then, only heat transfer through convection is worth considering, which is similar to boundary conditions in the flank and bottom,

$$-\lambda\frac{\partial T}{\partial z} = k_a(T - T_e), 0 < x < W, 0 < y < L, z = H \tag{13}$$

where k_a represents the convective heat transfer coefficient between water and air. As shown in Figure 5, the normal heat flux through the top is proportional to the temperature difference.

2.1.4 Absorbed Heat from Added Hot Water

For our bathtub, the only way it gets reheat is by hot water supply. As assumpted, hot water flow remains constant for each period. Denote Δm as the mass of hot water poured into the bathtub per unit time and T_h as its temperature. Then, the power of its equivalent point heat flux source can be expressed as

$$P = C\Delta m(T_h - T(x_e, y_e, z_e, t)) \tag{14}$$

and its emitting power density is

$$p = P\delta(x - x_e, y - y_e, z - z_e) = C\Delta m(T_h - T(x_e, y_e, z_e, t))\delta(x - x_e, y - y_e, z - z_e) \tag{15}$$

where $\delta(x, y, z)$ is the unit pulse function. Adding p to the RHS of Equation 4, we have

$$\rho C \frac{\partial T}{\partial t} = \lambda(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}) + C\Delta m(T_h - T(x_e, y_e, z_e, t))\delta(x - x_e, y - y_e, z - z_e) \quad (16)$$

which describes the inner thermodynamics including the effect of hot water.

2.2 Strategy Formulation

With the thermodynamics model in previous section established, we have all the knowledge of temperature distribution over space and time. Since both relaxation experience and water usage are under our consideration, we optimize our strategy in a two-phase manner. First, we formulate temperature requirements for bather experience. Then, based on these requirements, we provide a series of criteria for strategy formulation and discuss the choices of water pouring rate Δm and temperature T_h.

2.2.1 Temperature Requirements

The foremost requirement of a strategy is to provide satisfactory experience for bather. For temperature, not only should the average temperature in the bathtub fall in the acceptance level of bathing, the variation of temperature over the bathtub should also be taken into consideration. Under this rationale, we provide the following requirement for temperature,

$$\overline{T}(t) + \alpha \Delta T(t) \leqslant T_{max}, \text{when adding water} \quad (17)$$

and

$$\overline{T}(t) - \alpha \Delta T(t) \geqslant T_{min}, \text{otherwise} \quad (18)$$

where

$$\overline{T}(t) = \frac{1}{|V|} \iiint_V T(x, y, z, t) dV \quad (19)$$

is the average temperature at time t and

$$\Delta T(t) = \sqrt{\frac{1}{|V|} \iiint_V [T(x, y, z, t) - \overline{T}(t)]^2 dV} \quad (20)$$

is the standard deviation of temperature at time t. α is an individually dependent factor. Large α means the bather has strict requirement for even temeperature while small α indicates the bather is very tolerant of temperature difference in the bathtub. T_{min} and T_{max} are the acceptable temperature range of hot water immersion.

The conditions of these requirements can be explained as follows. When pouring hot water, the bather pays attention to not getting burned, while when no hot water is being add, the bather would be uncomfortable only when it's too cold in the bathtub. Note that although we have mathematic requirements as above, it is actually modeling the real feeling of the bather: when he or she is feeling hot, instead of feeling the the water is hot 'averagely', only part of water accounts for his feeling, so a $\alpha \Delta T$ is added to $\overline{T}(t)$ to emphasize that phenomenon. Same reason accounts for the $-\alpha \Delta T$ in the other requirement.

2.2.2 Greedy Strategies

Although an infinite set of strategies are theoretically possible, most of them are inefficient and impractical. A strategy requiring the bather to change the state of faucet frequently would totally compromise his or her mood, let alone open the facuet to an accurate given extent (e.g. 82.98%). Here, only a series of easy-to-handle and approximately optimal strategies are discussed.

Consider the temperature requirements expressed in Equation 17 and Equation 18. The time of adding hot water cannot be later than the moment when equality holds in Equation 18, that is, when the bather feels too cold. Similarly, the time of stopping cannot be later than the moment when equality holds in Equation 17, when it's too hot.

Starting prematurely (that is, prior to the latest possible time) is uneconomical. Compared to the latest occassion, averagely hotter water is drained out due to overflow, and average temperature is higher later on. Both of them leadd to more heat released to environment (recall in Newton's law of cooling, higher difference in temperature results in higher heat loss rate), and eventually, lead to an earlier time of reheating, which requires more water for a given period of bathing.

Stopping prematurely, however, may lead to economical results. Consider, for example, the bather is going to leave the bathtub in several seconds, and the heat in the water is already enough to satisfy the bather until he finishes. Continuing pouring water is of course a waste, but such occassion is hardly predictable in practical, especially if the faucet is controlled by the bather rather than a powerful computer system. In addition, this rationale also leads to fewer number of faucet operations.

Therefore, our **first criterion** is: *Start adding hot water when it's too cold, and stop it when it's too hot.* The threshold are given by Equation 17 and Equation 18, when equality holds.

The second criterion is about the choice of possible rate of pouring, Δm. As mentioned earlier, it is unrealisitc to specify an exact figure. For an given faucet, the maximum water flow through it is denoted as Δm_{max}. Our **second criterion** is: *The rate of pouring, Δm, should be one value in the set* $\{0, 0.25\Delta m_{max}, 0.5\Delta m_{max}, 0.75\Delta m_{max}, \Delta m_{max}\}$.

With the choices of starting and stopping time and Δm determined, the last issue is the hot water temperature T_h. T_h itself should first fall in $[T_{min}, T_{max}]$. Then, the exact choice of T_h is stated as in our **third criterion**: *Hot water temperature $T_h \in [T_{min}, T_{max}]$ is chosen as such that minimum amount of total water is used.* The total mount water W_{total}, is expressed as

$$W_{total} = \sum_{i=1}^{N} \Delta m_i \Delta t_i \tag{21}$$

where N is the total number of periods of adding water during a bath, Δm_i is the pouring rate in ith period, and Δt_i is the length of that period.

Our **last criterion** is another practical consideration: *The total number of periods in a bath N should be no more than an accpetable upper limit depending on the time of bathing t_0.* The exact upper limit is also individually dependent. For example, we believe for a 10 minitutes' bath, only one period of adding hot water is acceptable.

2.3 Shapes, Postures and Additives

The geometric settings, shown in Figure 3 is an ideal approximation to the real-life situation. However, it can be easily modified to accommodate practical applications.

2.3.1 Bathtub Shapes

The cuboid shape is an approximation to several types of modern bathtub. The bathtub in Figure 2 can be modeled by a cuboid shape. However, an oval cylinder provides better approximation for modeling curvature in both ends. In our model, replacing the cuboid bathtub with an oval cylinder simply leads to different boundary conditions in the flank. The respective geomtric setting with boundary heat transfer is shown in Figure 6.

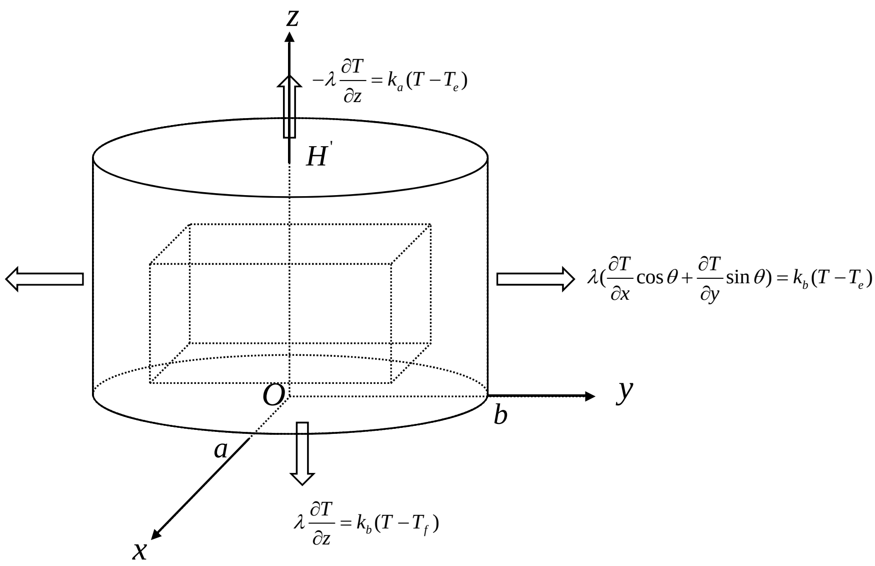

Figure 6: Geometric setting of oval bathtub with heat flows.

The modified boundary condition in the flank is

$$\lambda(\frac{\partial T}{\partial x}\cos\theta + \frac{\partial T}{\partial y}\sin\theta) = k_b(T - T_e), x = a\cos\theta, y = b\sin\theta, 0 \leqslant \theta \leqslant 2\pi, 0 < z < H' \quad (22)$$

while the boundary conditions for the top, bottom and between the bather and water remain unchanged.

Other shapes can be implemented in a similar manner.

2.3.2 Bather Postures

When resting in a bathtub, the bather can be expected to take several postures other than lying flat. Different postures can be implemented by setting different boundary conditions between water and the bather.

For example, consider the bather is sitting straight with his or her back on the right side of the tub and head above water. The geometric setting and heat flows are illustrated in . The updated boundary conditions are listed as follows.

Between water and the bather:

$$\frac{\partial T}{\partial x} = 0, x = x_b \pm \frac{w}{2}, L - l_1 < y < L, 0 < z < h \text{ or } x = x_b \pm \frac{w}{2}, L - h < y < L, h < z < H \tag{23}$$

$$\frac{\partial T}{\partial y} = 0, x_b - \frac{w}{2} < x < x_b + \frac{w}{2}, y = L - l_1, 0 < z < h$$
$$\text{or } x_b - \frac{w}{2} < x < x_b + \frac{w}{2}, y = L - h, h < z < H \tag{24}$$

$$\frac{\partial T}{\partial z} = 0, x_b - \frac{w}{2} < x < x_b + \frac{w}{2}, L - l_1 < y < L, z = 0$$
$$\text{or } x_b - \frac{w}{2} < x < x_b + \frac{w}{2}, L - l1 < y < L - h, z = h \tag{25}$$

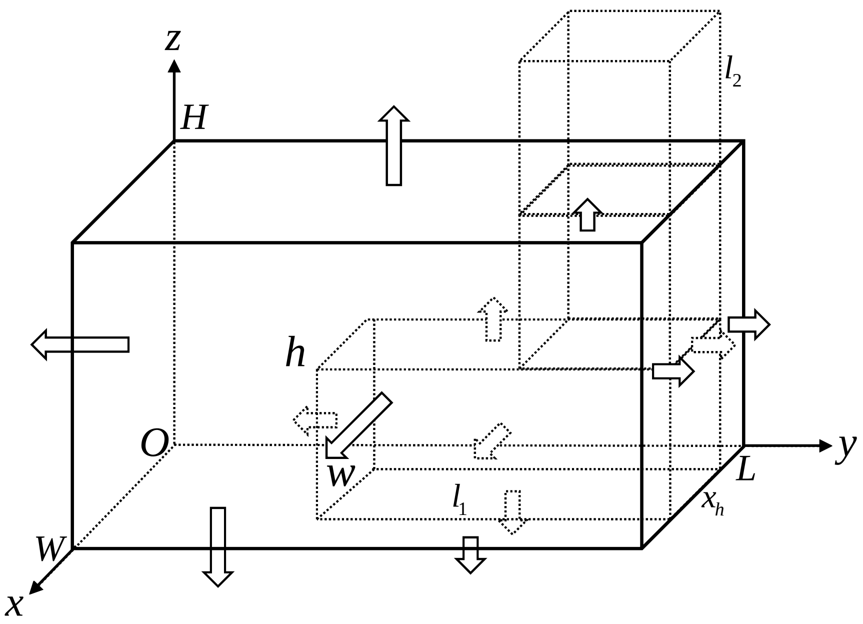

Figure 7: Geometric setting of sitting bather with heat flows. Solid arrows denotes heat transfer between water and environment. Dash arrows denotes heat transfer between water and the bather (and in fact, it is insulated).

Between water and environment:

$$\lambda \frac{\partial T}{\partial x} = k_b(T - T_e), x = 0, 0 < y < L, 0 < z < H \tag{26}$$

$$-\lambda \frac{\partial T}{\partial x} = k_b(T - T_e), x = W, 0 < y < L, 0 < z < H \tag{27}$$

$$\lambda \frac{\partial T}{\partial y} = k_b(T - T_e), 0 < x < W, y = 0, 0 < z < H \tag{28}$$

$$-\lambda\frac{\partial T}{\partial y} = k_b(T-T_e), 0<x<x_b-\frac{w}{2}, y=L, 0<z<H$$
$$\text{or } x_b+\frac{w}{2}<x<W, y=L, 0<z<H \tag{29}$$
$$\lambda\frac{\partial T}{\partial z} = k_b(T-T_f), 0<x<W, 0<y<L-l_1, z=0$$
$$\text{or } 0<x<x_b-\frac{w}{2}, L-l_1<y<L, z=0 \text{ or } x_h+\frac{w}{2}<x<W, L-l_1<y<L, z=0 \tag{30}$$
$$-\lambda\frac{\partial T}{\partial z} = k_a(T-T_e), 0<x<W, 0<y<L-h, z=L$$
$$\text{or } 0<x<x_b-\frac{w}{2}, L-h<y<L, z=0 \text{ or } x_h+\frac{w}{2}<x<W, L-h<y<L, z=0 \tag{31}$$

2.3.3 Bubble Additives

Adding additives for a layer of surfactant foam on the surface of the water is a common practice for bathers. The additives can help cleanse the skin while the foam insultates the bath water, keeping it warm for a longer time. In our model, the effect of the layer of foam is illustrated by a different coefficient of convective heat transfer in the top. k_f, the convective heat transfer coefficient between water and foam replaces k_a, that between water and air. The resulting boundary condition is

$$-\lambda\frac{\partial T}{\partial z} = k_f(T-T_e), 0<x<W, 0<y<L, z=H \tag{32}$$

Different types of additives can generate foam layers with various k_f, though $k_f < k_a$ should be maintained in order to compromise heat loss.

3 Model Implementation and Results

4 Final Remarks

4.1 Strengths and weaknesses

We formulate the problem as a thermodynamics model and it is purposely designed to be customizable and applicable to a variety of scenarios. This leads to a set of strengths and weakenesses.

Strenths:

- **Exact temperature field for better considerations.** Our model presents an exact temperature field over space at any moment rather than the average temperature only. This facilitates formulating a more considerate strategy for bathers by taking other statistics into account. For example, we guarantee the temperature difference fall in an acceptable range through the introduction of standard deviation of temperature over the bath.

- **Applies widely.** Our model is capable of being used in most bathing occassions by incorporating individually specific factors. Different shapes of bathtub, bather's postures and formulae of additives can be easily considered. In addition, individual tolerance of temperature difference can be varied to cater to personal preference.
- **Accommodates to practical situations.** We establish a series of criteria to filter out possible strategies. In this way, our optimal strategies are ensured to yield both economical and applicable results.

Weakenesses:

- **Computational efforts.** To obtain the accurate temperature distribution over both space and time, we solve the thermodynamics partial differential equations numerically with finite difference methods. Though yielding promising temperature filed, this method is computationally expensive.
- **Idealized situation** Several idealized assumptions are used in an effort to maintain a solvable model. Minor factors are neglected and may result in deviations.

4.2 Future Development

Most of the issues arise from that some assumtions are over-simplyfied and unique factors of the scenario are overlooked. In the future, we would further develop and optimize our model in the following ways:

- **Extend the model.** Our model only considers the duration when the tub is already full. An extended model including the time before may be helpful.
- **Account for evaporation process.** The evaporation is assumed at equilibrium in our model. Its dynamics can be model in a more accurate way.
- **Consider a dynamic foam layer.** Since water loss through water drain can take away foam, a dynamic foam layer based on the amount of water already added is more realisitic.
- **Rework the model of the bather.**. Heat transfer between bather's body and water invovles biodynamics of human. Incorporating it into our model leads to more faithful results.

4.3 Conclusion

We model the optimal strategy of adding hot water into a bathtub in practical situations. Firstly, we establish the thermodynamics of bathtub water with heat transfer equations. Based on that, strategies considering both water usage and bather's experience, including requirements average temperature as well as temperature difference, are formulated. These strategies follow a series of criteria that are especially tailored to be considerate for bathers. Then, we further extend our model by discussing the ways of accommodating to different bathtub shapes, bather postures and additives. The numeric experiments show promising results and illustrate the applicablity of our strategies.

References

[1] Garrett G Fagan. *Bathing in public in the Roman world*. University of Michigan Press, 2002.

[2] Bath & North East Somerset Council 2017. The roman bath, one of the finest historic sites in northern europe, January 2017. Retrieved from https://www.romanbaths.co.uk/.

[3] Bathtub Refinishers. Bathtub refinishing, January 2017. Retrieved from http://www.bathtub-refinishers.com/bathtub-refinishing.

[4] Rana A Fine and Frank J Millero. Compressibility of water as a function of temperature and pressure. *The Journal of Chemical Physics*, 59(10):5529–5536, 1973.

[5] Adrian Bejan. *Convection heat transfer*. John wiley & sons, 2013.

[6] Isaac Newton. *Isaaci Newtoni Opera quae exstant omnia*, volume 4. excudebat Joannes Nichols, 1779.

Future Technology*:

Independent Mathematical Consulting Services

Future Technology, L.L.C.*
Letter to World Medical Association
1/23/17

To whom it may concern:

To keep the bath warm, either a secondary heating system must be present or hot water must be added once in a while. In the latter case, when and how much hot water is added to the bathtub is of importance. Insufficient water supply leads to uncomfortable experience, while excessive amount of water goes wasted. In addition, the temperature of newly added water should be chosen wisely. Since hot water is often added from a faucet in the end of the bathtub, it takes time for the temperature turns even in the bath through heat transfer. Pouring extreme hot water generally saves total amount of water used, but large temperature difference impedes relaxation and can even cause burns.

In this report, our primary objective is to find the best strategy of adding hot water to a bathtub such that minimum amount of water is wasted through overflow drain while both the average temperature and temperature difference are controlled in an acceptable range. To accomplish this, we first establish a general model of temperature of bathtub under an ideal condition, with a cuboid bathtub shape, a cuboid bather shape and no bather motions involved. A series of greedy-based practical strategies are then formulated. Different bathtub shapes and bather postures, as well as bath addictive are further discussed to accommodate to a real-life situation. For each specific condition, its optimal strategy is highlighted.

Starting prematurely (that is, prior to the latest possible time) is uneconomical. Compared to the latest occasion, averagely hotter water is drained out due to overflow, and average temperature is higher later on. Both of them leads to more heat released to environment (recall in Newton's law of cooling, higher difference in temperature results in higher heat loss rate), and eventually, lead to an earlier time of reheating, which requires more water for a given period of bathing.

Stopping prematurely, however, may lead to economical results. Consider, for example, the bather is going to leave the bathtub in several seconds, and the heat in the water is already enough to satisfy the bather until he finishes. Continuing pouring water is of course a waste, but such occasion is hardly predictable in practical, especially if the faucet is controlled by the bather rather than a powerful computer system.

*This company is purely a work of fiction.