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2017 Mathematical Contest in Modeling (MCM) Summary Sheet
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Break through the Bottleneck: Optimizing Designs for Toll Plazas with a Cellular-automation Approach

Summary

Barrier tolls in highways can serve as bottlenecks, restricting the maximum rate of traffic flow. By appropriately designing the toll plaza, its negative effect can be relieved, reducing the probability of a congestion. Quantifying the optimal design of the merging area in a toll plaza is the subject of this report.

First, we establish a cellular-automation model to simulate the traffic flow of the fan-in area. Based on several key assumptions about drivers, we invent a set of rules to govern the behaviors of cars. Cars arriving at booths are delayed and slowed down according to the type of booth. After entering the merging area, they aggressively change their speed as long as safety conditions are satisfied. Lane-changing strategy is to move towards regular travel lanes and to seek opportunities of acceleration if they are already in.

Next, a general metric, Composite Performance Index (CPI), is formulated to evaluate the performance of a merging area design. To incorporate cost, accident prevention and throughput into CPI, we adopt three dimensionless scenario-independent metrics: land utilization ratio, hard brake ratio and throughput ratio as respective criteria.

We then implement our model with different shapes and merging patterns. Simulation results show that for a eight-to-four fan-in scheme, our purposely designed *double-slant* toll plaza, with approximately *1450.1* square yards' coverage, outperforms traditional rectangular plazas and single-slant plazas. Additional merging patterns, apart from the basic *move-towards-travel-lane* pattern introduced in lane-changing rule, has little impact, if any, on performance.

Finally, we conduct sensitivity analysis to further explore the performance of our design in different scenarios, including various traffic conditions, proportions of booths and introduction of autonomous vehicles. The results show that these parameters are insensitive to our optimal design.

Contents

1 Introduction

1.1 Background

Traffic congestion is almost unavoidable in New Jersey during rush hours. As is shown in the 2015 Urban Mobility Scorecard [1], New Jersey drivers spend up to two weeks a year stuck in jams, marking it some of the worst traffic in the nation. As a matter of fact, even in off-peak hours, it is still common practice to experience long delays at a toll plaza on, for example, the Garden State Parkway. While adding additional lanes on major roads would be logistically impossible [2], redesigning the toll plaza so that it can work in a more efficient manner, seems to be a promising solution.

For a toll plaza, the optimal number of tollbooths is well studied in 2005 MCM (see [3], [4], and [5], for example). In addition, based on toll plaza circulation safety analysis, the optimal tollbooth type placements under different traffic composition scenarios given the total number of tollbooths, are also determined [6]. However, the optimal design of the merging area, where vehicles fan in from B tollbooth egress lanes down to L lanes of traffic ($B > L$), still remains unknown.

1.2 Our Work

In this report, our primary objective is to find the best design of the merging area, such that accident prevention, high throughput and low cost. To accomplish this, we establish a basic cellular-automation model of traffic flow.

For that model to work properly in our case, where merging patterns, shape and size of merging area are to be determined, additional rules should be given. We first provide the rule of leaving cars from the booths, based on the exponential distribution of their interarrival times and delay. Then, speed update rule is formulated to allow the drivers aggressively gain speed while avoiding collision. After that, we develop the lane-changing rule based on the fact that cars change lanes if they can get closer to regular travel lanes when they are in tollbooth egress lanes, or if they can accelerate when they are already in regular ones. Finally, positions of cars are updated according to their speed and lane-changing scheme.

With our cellular-automation model of traffic established, we formulate the metrics of performance of a toll plaza design. We apply three dimensionless scenario-independent measures: land utilization ratio, hard brake ratio and throughput ratio, to evaluate the cost, accident prevention and throughput respectively. Composite Performance Index (CPI), is then formulated as a weighted sum of these metrics.

We then implement our model with different shapes. Two conventional shapes: rectangular plaza and front-step plaza and two purposely designed shapes: back-step plaza and double-step plaza are considered. The effect of an additional merging pattern: prohibit any cars from travelling between automatic booth lanes and electronic booth lanes, is analyzed.

Finally, different traffic conditions, proportions of booths and introduction of autonomous vehicles are discussed, by changing entering parameters of model and modifying relative rules.

1.3 Nomenclature

The nomenclature used in this report is listed as follows:

Symbol	Definition
B	Number of tollbooths in each direction.
L	Number of lanes in each direction.
$(x(t), y(t))$	Position of car in units of step sizes Δx and Δy at time t.
Δx	Step size along the traffic flow, chosen to be $16ft$.
Δy	Step size perpendicular to the traffic flow, chosen to be $12ft$.
Δt	Increment of time, chosen to be $1s$.
λ	Cars arriving rate at the toll plaza uniformly.
$v_c(t)$	Speed of the current car at time t.
$v_{b'}(t)$	Speed of the back car in the target lane at time t.
gap_{cf}	Length of the gap between the current car and the car in front.
$gap_{c'f'}$	Length of the gap between the front car and the expected car position of the target lane.
$gap_{b'c'}$	Length of the gap between the back car and the expected car position of the target lane.
η_{land}	land utilization ratio of a design.
η_{brake}	Hard brake ratio of a design.
$\eta_{throughput}$	Throughout ratio of a design.
CPI	Composite Performance Index.
$\omega_1, \omega_2, \omega_3$	Weights of cost, accident prevention and throughput, respectively.

1.4 Assumptions

The reliability of our models depends on certain key, simplifying assumptions, which are listed below:

- **Cars arrive at the each tollbooth uniformly in time.** This indicates the interarrival times has an exponential distribution with arriving rate λ. A driver's decision to drive on an incoming lane at a particular time is independent from that of any other driver. And since our focus is the merging area, the 'fanning in' scheme is of little interest. Thus for simplicity, we assume each driver randomly chooses a booth, which leads to the assumption above.

- **Drivers are greedy.** Drivers always want to keep the highest speed as they can, as long as collisions can be avoided. That is, they only slow down when it is a must.
- **It takes no time to change lanes.** The time cost is considered to be compensated by acceleration by drivers, which is a common practice when changing lanes. In addition, in a single time slot, cars can only change to its adjacent lanes.
- **Tollbooths are categorized into electronic booths, exact-changed booths and human-staffed booths.** Cars coming through an electronic booth are delayed $1s$ only, and restart with the maximal speed that avoids collisions. Cars passing through an exact-changed booth or a human-staffed booth, however, restart with no initial speed since they must stop at the booth. Delays for these two booths are $16s$ and $10s$, respectively.
- **Steady acceleration and hard brake.** The acceleration is approximately constant, as of most cars. To avoid collision, drivers can take hard brake, decelerate immediately to a safe speed.

2 Model Theory

2.1 Cellular-automation Model

The cellular-automation method is a common approach to simulate traffic flow. First proposed by Nagel and Schreckenberg [7], this method views both time and space in a discrete manner.

The space consists of a regular grid of cells, each one either occupied by a car or empty. As illustrated in Figure 1, cars move from left to right (forward along x-axis) in a three-lane highway. The position of each cell is represented by the coordinates $(x(t), y(t))$ of its top right corner, in units of step sizes Δx and Δy. The step size along x-axis, Δx is chosen to be $16ft$, approximately the length of a SUV, while $\Delta y = 12ft$, the standard lane-width specified by Interstate Highway standards in the US.

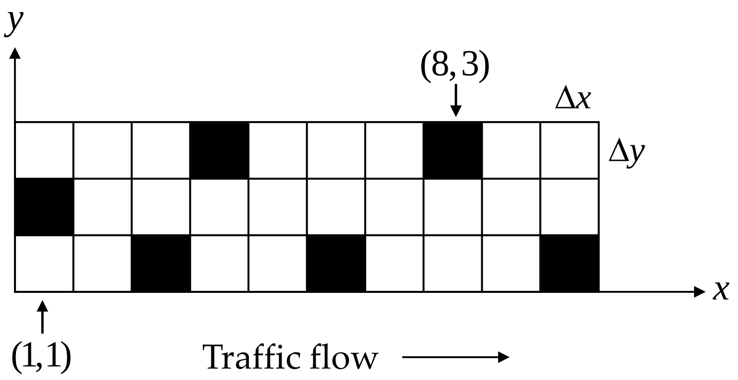


Figure 1: The cellular-automation representation of a three-lane highway at a given time slot. Black rectangular denotes a car.

The time is discretized by a small interval Δt, which we choose to be $1s$. After each interval (from t=$n\Delta t$ to $t = (n+1)\Delta t$), the status of all cells are updated based on a set of rules. All cars move forward based on his speed and lane-changing strategy. New cars enter the area based on a random distribution. The details of the rule set are introduced in the following subsections.

2.2 Cars Entering Rule

As assumed, the interarrival times is an exponential distribution with arriving rate λ, and the arrival time between different lanes are independent from each other. Therefore, for each time slot, cars arrive at bth booth with a probability $\lambda\Delta t$. To implement the random process of car arriving, we adopt the transformation method for determination of whether a car enters [8]. We first generate a uniform random number in $[0, 1]$ ϵ. If $\epsilon < \lambda\Delta t$, a car arrives at ith booth. Otherwise, no car arrives in this time slot. When a car arrives at a free booth, it begins its delay according to the type of booth. If the booth is busy, the car instead enters a queue waiting. When a car finishes its delay, it leaves from the booth with either an initial speed (electronic booth) or none (exact-changed or human-staff booth). Then, if the queue is not empty, the first car in it enters the booth. The rule is summarized in Algorithm 1.

Algorithm 1 Cars Entering Rule

```
function CarEntering
    for each booth $b$ do
        Generate a random number $\epsilon \in [0, 1]$
        if $\epsilon < \lambda\Delta t$ then
            if the queue is empty then
                The car enters the booth, initialize a $timer_b$
            else
                The car enters the queue
            end if
        end if
        Update $timer_b$ if any.
        if $timer_b = delay_b$ then
            The car in the booth leaves and enters the merging area
            if the queue is not empty then
                The first car in queue enters the booth, initialize a $timer_b$
            end if
        end if
    end for
end function
```

2.3 Speed Update Rule

Drivers are expected to keep high speed as possible as they can, as long as collision can be avoided. The maximum speed in New Jersey is $65mph$, approximately $6cell/increment$. The acceleration is a constant, $1cell/increment^2$ in our case. To avoid collision, a driver must ensure that even if the car in front suddenly stops, it can still use next time slot for a hard brake. Thus,

$$v_c(t+\Delta t) = gap_{cf}/\Delta t, \text{ if } v_c(t)\Delta t > gap_{cf} \tag{1}$$

where $v_c(t)$ is the speed of the car at time t, and gap_{cf} is the length of gap between the current car and the car in front. However, the above rule based on the condition that drivers estimate exactly the distance and the speed, as well as act immediately without reaction time. In practice, drivers turn to keep an extra distance. Here, we choose this distance to be $\lfloor v_c(t)/2 \rfloor \Delta t$. This makes it possible for cars to tailgate at low speed ($v_c(t) = 1$) and aggressively accelerate with no cars in sight. The modified update rule is

$$v_c(t+\Delta t) = \max\{0, gap_{cf}/\Delta t - \lfloor v_c(t)/2 \rfloor\}, if\ v_c(t)\Delta t + \lfloor v_c(t)/2 \rfloor \Delta t > gap_{cf} \tag{2}$$

If the gap is long enough to avoid collision with current speed, the driver would seek acceleration, as long as it doesn't reach the speed limit, that is

$$v_c(t+\Delta t) = \min\{6, v_c(t)+1\}, \text{ if } v_c(t)\Delta t + \lfloor v_c(t)/2 \rfloor \Delta t < gap_{cf} \tag{3}$$

In other case, the car would remain here current speed. The complete speed update rule is summarized in Algorithm 2. Note that cars entering the merging area from electronic booths have initial speed, which is set to be the maximal possible speed to avoid collision,

$$v_c(t)\Delta t + \lfloor v_c(t)/2 \rfloor \Delta t > gap_{cf} \tag{4}$$

In our model, the dead end of toll plaza is also considered as a special 'car' with fix position, no speed and no status update. Therefore, it will also be used for the calculation of gap_{cf} and $gap_{c'f'}$.

Algorithm 2 Speed Update Rule

```
function SPEEDUPDATE($v_c(t)$,$gap_{cf}$)
    if $v_c(t) + \lfloor v_c(t)/2 \rfloor > gap_{cf}/\Delta t$ then
        $v(t+\Delta t) \leftarrow \max\{0, gap_{cf}/\Delta t - \lfloor v_c(t)/2 \rfloor\}$
    end if
    if $v_c(t) < gap_{cf}/\Delta t$ then
        $v_c(t+\Delta t) \leftarrow \min\{6, v_c(t)+1\}$
    end if
    if $v_c(t) + \lfloor v_c(t)/2 \rfloor = gap_{cf}/\Delta t$ then
        $v_c(t+\Delta t) \leftarrow v_c(t)$
    end if
    return $v_c(t+\Delta t)$
end function
```

2.4 Lane-changing Rule

In a regular travel lane, drivers considering lane-changing only when they can accelerate. By estimating the speed of the back car in the target lane, $v_{b'}$ and the gap $gap_{b'c'}$, as show in Figure 2, the driver can ensure that the back car wouldn't collide with his or her car, even if he or she takes an immediate stop. This requires

$$gap_{b'c'} > (v_{b'} + \lfloor v_{b'}/2 \rfloor + 1)\Delta t \tag{5}$$

where the $\lfloor v_{b'}/2 \rfloor$ term is also for safety concern and $+1$ term because the driver has no idea whether the back car would accelerate. In addition, the acceleration opportunity requires

$$gap_{cf} \leqslant (v_c(t) + \lfloor v_c/2 \rfloor)/\Delta t < gap_{c'f'} \tag{6}$$

That is, the car can't accelerate in the current lane but can gain speed if it moves to the target lane. When both adjacent lanes can leads to acceleration, the lane with larger $gap_{c'f'}$ is chosen to be target. And the speed update should also consider the fact that if the lane-changing is applied, gap_{cf} should be replaced with $gap_{c'f'}$ to keep safe.

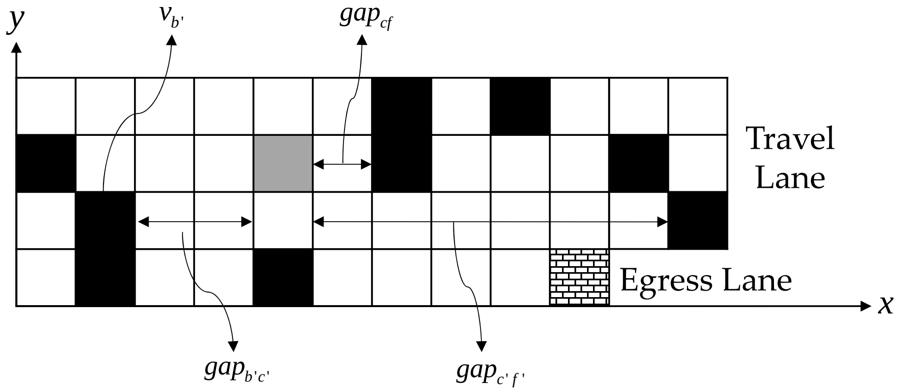


Figure 2: The illustration of lane changing in a regular travel lane. Shadow cell denotes the current car under consideration. Wall cells denote the end of egress lanes. Note that the current car would consider both adjacent lanes for changing.

If the car is currently in an egress lane, however, the acceleration is a minor priority. Top consideration is to change to lanes that are regular travel lanes or closer to them. Therefore, the driver would take every opportunity to change, regardless of acceleration of deceleration. This condition is shown in Figure 3. In addition, the car would never change to lanes that are farther from regular travel lanes.

The last lane-changing rule worth mentioning is that a car already in a regular travel lane would never try to change to a tollbooth egress, even though it may lead to acceleration opportunity. This is also very rational since egress lanes generally mean more traffic and finally, the car still needs to change back. The complete lane-changing rule is summarized in Algorithm 3.

2.5 Position Update Rule

We update the position of each car at time t according to its speed at $t + \Delta t$. That is,

$$x(t + \Delta t) = x(t) + v(t + \Delta t) \tag{7}$$

If lane-changing is required, we also alter its y coordinate according to the choice of the target lane,

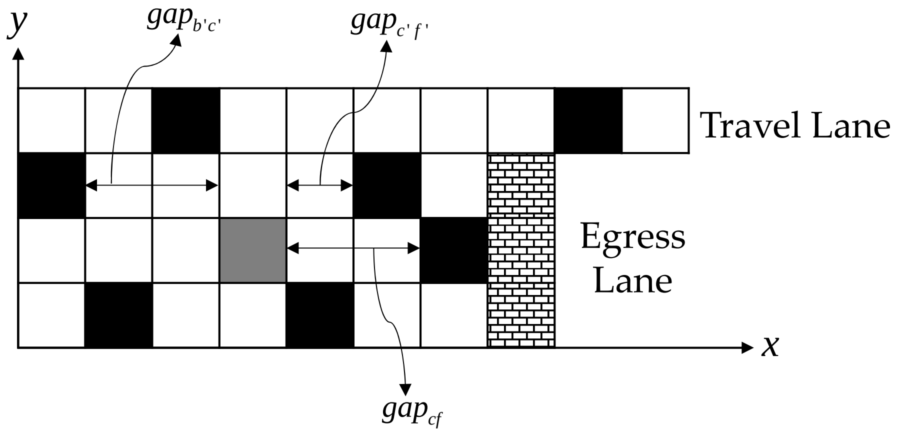


Figure 3: The illustration of lane changing in an egress lane. Shadow cell denotes the current car under consideration. Wall cells denote the ends of egress lanes. Note that the current car would only consider changing to the right lane since it's closer to the regular travel lane.

Algorithm 3 Lane-changing Rule

```
function LANECHANGING($LaneType$, $v_c(t)$, $gap_{cf}$, $gap_{c'f'}$, $gap_{b'c'}$, $v_{b'}$)
    $LaneChanged = False$
    if $LaneType = EgressLane$ then
        if $gap_{b'c'} > (v_{b'} + \lfloor v_{b'}/2 \rfloor + 1)\Delta t$ then
            $LaneChanged = True$
        end if
    else
        if $gap_{b'c'} > (v_{b'} + \lfloor v_{b'}/2 \rfloor + 1)\Delta t$ && $gap_{cf} \leqslant (v_c(t) + \lfloor v_c(t)/2 \rfloor)\Delta t < gap_{c'f'}$ then
            $LaneChanged = True$
        end if
    end if
    return $LaneChanged$
end function
```

$$y(t+\Delta t) = y(t) \pm 1, \text{ depending on the target lane} \tag{8}$$

The position update rule is shown in Algorithm 4. With this final rule introduced, our rule set for the cellular-automation model is finalized. The complete procedure in shown in Algorithm 5. Note that we take a synchronous update to ensure that when each car is considered, the status (position and speed) of other cars that may make a difference to the results (e.g. the car in the front) still remain unchanged.

Algorithm 4 Position Update Rule

```
function PositionUpdate($x(t), y(t), v(t+\Delta t), LaneChanged$)
    $x(t+\Delta t) \leftarrow x(t) + v(t+\Delta t)$
    if $LaneChanged$ then
        $y(t+\Delta t) \leftarrow y(t) \pm 1$ depending on the target lane
    else
        $y(t+\Delta t) \leftarrow y(t)$
    end if
    return ($x(t+\Delta t)$,$y(t+\Delta t)$)
end function
```

Algorithm 5 The Cellular-automation Model

```
for $time = 0 \rightarrow N\Delta t$ do
    CarEntering
    for all cars do
        Obtain its $x(t)$, $y(t)$, $v(t)$, $gap_{cf}$, $gap_{c'f'}$, $gap_{b'c'}$, $v_{b'}(t)$, $LaneType$
        $LaneChanged \leftarrow$LaneChanging($LaneType$,$v_c(t)$,$gap_{cf}$,$gap_{c'f'}$,$gap_{b'c'}$,$v_{b'}(t)$)
        if $LaneChanged$ then
            $gap_{cf} \leftarrow gap_{c'f'}$
        end if
        $v_c(t+\Delta t) \leftarrow$SpeedUpdate($v_c(t)$,$gap_{cf}$)
        $(x(t+\Delta t), y(t+\Delta t)) \leftarrow$PositionUpdate($x(t), y(t), v_c(t+\Delta t), LaneChanged$)
    end for
    for all cars do
        $(x(t), y(t)) \leftarrow (x(t+\Delta t), y(t+\Delta t))$
    end for
end for
```

2.6 Metrics of Performance

The performance of a toll plaza design incorporates several factors. Startup cost including land purchasing and road construction determines whether the design is economically applicable. Safety concerns involve the frequency when drivers are forced to take dangerous actions, for example, hard braking. Throughput, as the most effective indicator of a toll plaza, should also be taken into account.

However, directly combining these factors together is troublesome. They have different dimensions and each involves several scenario-specific parameters. For instance, the land price for different states can vary dramatically and throughput is positively correlated to the number of travel lanes and egress lanes. Therefore, we adopt three dimensionless and occasion-independent ratios as the metrics of performance.

The first metric is the land utilization ratio η_{land} that is defined by the area of the fan-in plaza over the available area.

$$\eta_{land} = \frac{S_{plaza}}{S_{available}} \tag{9}$$

The length (parallel to traffic flow) of the toll plaza is often given in prior, $160ft = 10cell$ in our case. This metric evaluates the land utilization efficiency and we use it to represent the general 'cost' of a design. Small value means less land used, and thus less cost required. The graphic illustration is shown in Figure 4.

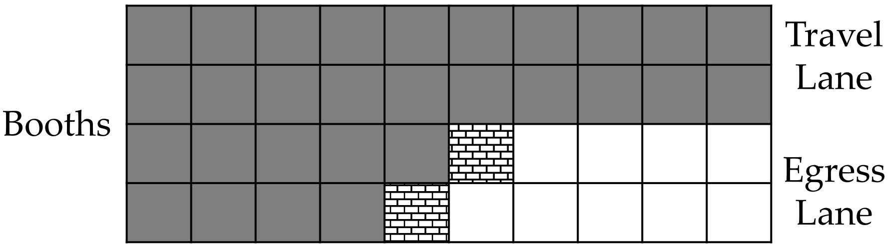


Figure 4: Graphic illustration of η_{land} of a design. Wall cells denote the ends of egress lanes. Shadow cells denote the coverage of the plaza. The available area is all of the cells in the figure.

The second metric is to evaluate the probability of an accident. Since we already assume that cars are capable of stopping immediately, regardless of their current speed, no accident 'happens' in our model. However, in practice, taking a hard brake doesn't guarantee immediate stop and accounts for most of accidents. Here, we take the average ratio of the number of hard brakes to number of cars over a certain period, η_{brake}, to evaluate the safety of a design,

$$\eta_{brake} = \frac{N_{hard\_brake}}{N_{cars}} \tag{10}$$

where a hard brake is defined as the speed changed in a time slot is greater than or equal to $2cell/increment^2$. Small value indicates low frequency of hard brake, and thus low probability of possible accidents.

To measure the throughput of a design, a naive notion would be the ratio of average leaving rate of the toll plaza to average entering rate from booths over a certain period. However, considering the delayed effect of booths, the maximal entering rate is limited to a very small value, 1 car per 10 seconds for an automatic booth, as in our case, regardless the traffic conditions. Any normal design of a merging area can handle so low a traffic with ease. Therefore, to better illustrate the actual capability of handling traffic, we denote the throughput ratio, as the average speed of leaving cars over maximal speed, $6cell/increment$,

$$\eta_{throughput} = \frac{v_{ave}}{v_{max}} \tag{11}$$

Compared to average speed itself, this ratio allows us to compare the capability of traffic handling between different plazas that adopts different speed limits. It is noteworthy that small value of $\eta_t hroughput$ means poor capability, thus less desired.

Finally, we define our Composite Performance Index (CPI), as a weighted composition of η_{land}, η_{brake} and $\eta_{throughput}$:

$$CPI = \omega_1 \eta_{land} + \omega_2 \eta_{brake} + \omega_3 (1 - \eta_{throughput}), \omega_1, \omega_2, \omega_3 > 0 \tag{12}$$

Note that $1 - \eta_{throughput}$, instead of $\eta_{throughput}$, is used in CPI. The reason is that both η_{land} and η_{brake} are negatively correlated with performance while $\eta_{throughput}$ is the opposite. Therefore, high value of CPI indicates poor performance of a design, and vice versa. By varying ω_1, ω_2 and ω_3, different priorities of cost, accident prevention and throughput can be easily considered. Our objective is to find the best design of toll plaza that minimizes CPI.

3 Model Implementation and Results

3.1 Toll Plaza Shapes

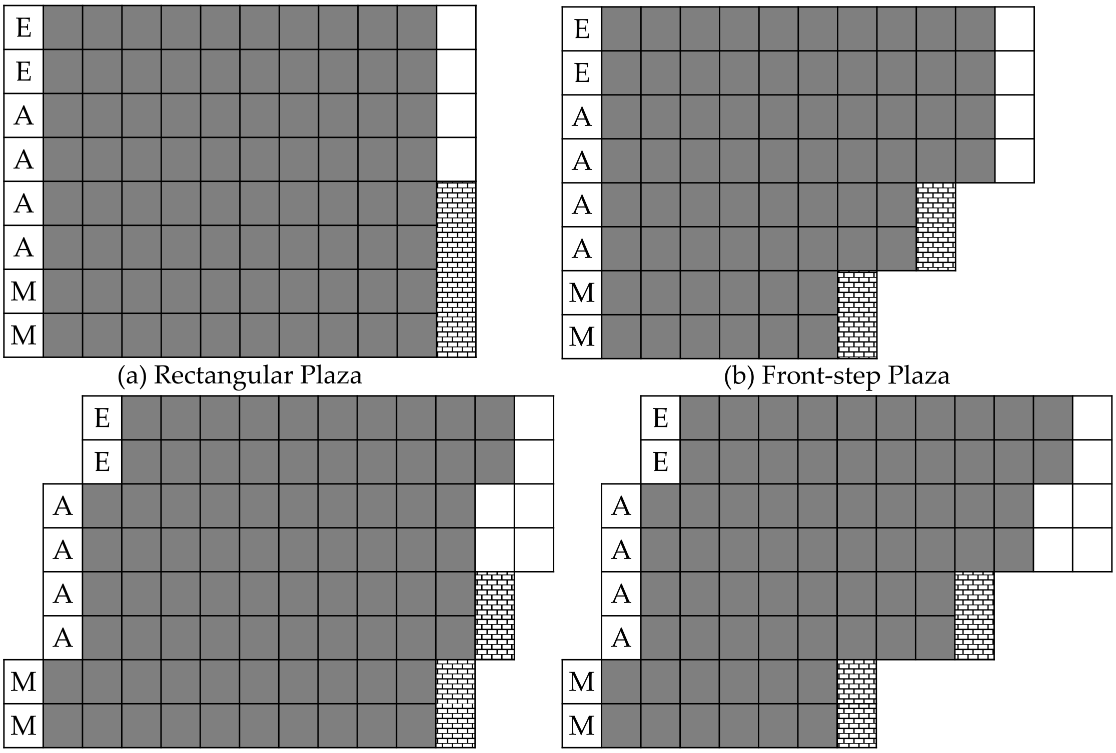
Prior to the detailed discussion of merging area design, toll booths placement should be considered. Taking the perspective of facilitate fanning out process, we place electronic booths in the innermost lanes of a plaza, followed by automatic and human-staffed ones. The reason is that electronic booths has the minimum delay and is expected to carry most of the traffic while human-staffed booths are much less efficient. In addition, cars don't need to slow down when passing an electronic booth and can leave the plaza quickly without deceleration to change lanes if they are already in a normal travel one. Cars passing through automated and human-staffed booths, however, don't have an initial speed and are suitable for lane-changing. [9] also suggests this placement based on safety considerations.

Now we begin our discussion about the shapes of merging area. Shown in Figure 5 are four eight-to-four fan-out designs under consideration. Both (a) and (b) are common designs, with a throughput priority and trade-off between throughput and cost. They represent rectangular plazas and single-slant plaza in practice. (c) and (d) are shifted version of (a) and (b), purposely designed by us to handle the traffic with tricky. The rationale behind these seemingly strange shapes is that by shifting electronic and automatic booths two and one cells forward, cars from outer lanes can move to inner lanes with relatively higher speed since they can accelerate in advance. This reduces the probability of blocking back cars leaving from outer booths due to low speed, especially when cars move from automatic booth egress lanes to electronic ones.

The available area of these designs are 80 cells as there are 8 lanes. Both (a) and (c) use all of them. (b) and (d) save 12 cells by shortening the length of egress lanes. Their respective land utilization ratio is

$$\eta^a_{land} = \eta^c_{land} = 1, \eta^b_{land} = \eta^d_{land} = 0.825 \tag{13}$$

To obtain the field performance metrics, we simulate these four designs with our



(a) Rectangular Plaza (b) Front-step Plaza

(c) Back-step Plaza (d) Double-step Plaza

Figure 5: Four shapes of merging area under consideration. Shadow cells denote the coverage of the plaza. The leftmost row indicates the type of booths: 'E' denotes electronic booths, 'A' denotes automated (exact-change) booths and 'M' denotes human-staffed (conventional) booth. The number and proportions of booths are suggested by [9].

cellular-automation model implemented in MATLAB. The traffic level is set as normal, with 2800 cars passing through the toll plaza each hour.

We set simulation period as 60 mins. The average hard braking ratio and average throughput ratio over time is shown in Figure 6. We first note that both hard brake ratio and throughput ratio are approximately constant over time, except for small fluctuations, showing the traffic flow is in a steady state during simulation. This allows us to use the average ratios over the simulation periods to represent the actual characteristics.

For hard brake ratio, back-step plaza and double-step plaza outperform rectangular plaza and to an enormous extent. This fact corresponds with our rationale of these designs. Frequency of hard brakes, mainly due to blocking low-speed cars moving from outer lines, are dramatically decreased. The difference between rectangular plaza and front-step plaza, between back-step plaza and double-step plaza, however, is neglectable in value. The reason is based on our observation that cars hardly move to the cells that are included in rectangular plaza and back-step plaza but excluded in front-step plaza and double-step plaza.

For throughput ratio, double-step plaza and back-step plaza share a similar performance, superior to rectangular plaza and front-step plaza, as expected. The reason of cars in a front-step plaza falling behind may be the trouble of gaining speed when changing lanes. Their counterparts in a double-step plaza, however, can accelerate in

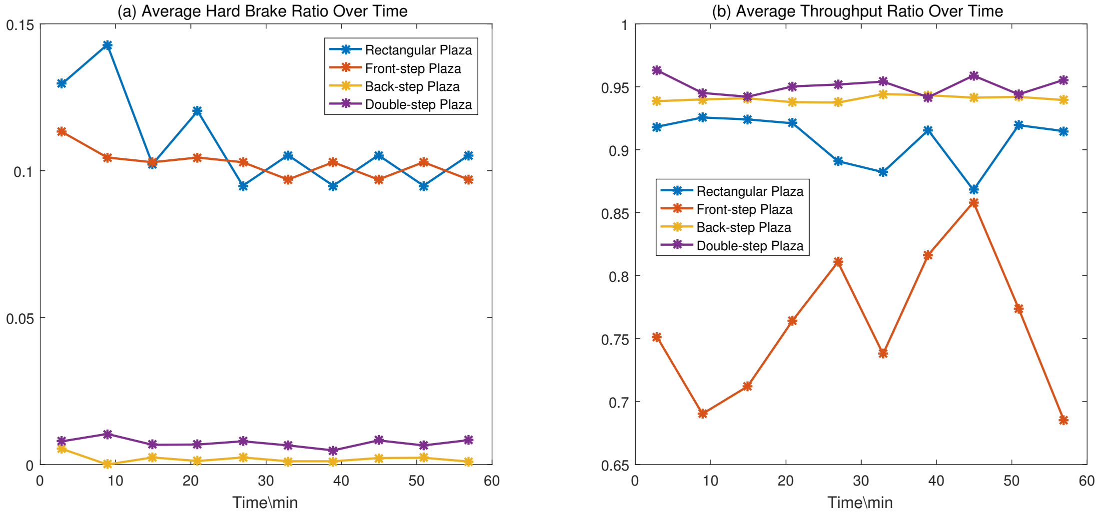


Figure 6: Comparison between four designs. (a) Average hard braking ratio and (b) average throughput ratio over time. Ten samples points marked by asterisk denote average values per 10 mins.

advance.

An important observation worth mentioning is that no traffic congestion occurs in all of the designs, as the cars leaving the merging area has a minimal average throughput ratio over 0.65, in a front-step plaza. This corresponds to our analysis when we develop $\eta_{throughput}$ in Subsection 2.6.

Since no prior information of the three ratios are given, the relative importance of the criteria is hard to determine. We take equal weights, $\omega_1 = \omega_2 = \omega_3 = 1$, for simplicity. The values of three ratios and CPI for these four designs are listed in Table 1. Best value of each dimension is highlighted.

Table 1: Land utilization ratio, hard brake ratio, throughput ratio and CPI of four plaza designs.

	η_{land}	η_{brake}	$\eta_{throughput}$	CPI
Rectangular Plaza	1	0.10956	0.90799	1.20157
Front-step Plaza	**0.825**	0.10248	0.76007	1.16741
Back-step Plaza	1	**0.0019**	0.94052	1.06138
Double-step Plaza	**0.825**	0.0074	**0.95065**	**0.88175**

As is obviously noted, the double-step plaza, which is purposely designed by us to facilitate traffic flow, ranks best in η_{land} and $\eta_{throughput}$, and has similar η_{brake}. Its $CPI = 0.88175$, far less than other designs, indicating it has a much better composite performance. Therefore, it is chosen as our optimal design that outperforms those in common use. The analysis of following subsections is based on this design.

3.2 Merging Patterns

To further exploit the potential of toll plaza, we consider set more regulations about lane-changing, that is, merging patterns. It is noteworthy that we already implement one basic merging patterns in Subsection 2.4, when we develop rules of changing lanes. We prohibit any cars from moving towards egress lanes if they are still in one (move-towards-travel-lane). This merging pattern is so rational that we think it 'basic'. Here, we consider another merging pattern: prohibit any cars from travelling between automatic booth lanes and electronic booth lanes in toll plaza. The rationale is that we don't want low-speed cars to occupy fast electronic booth lanes and hinders back cars.

Simulation results of double-step plaza with or without this additional merging pattern are shown in Figure 7. As we have noted, this additional merging pattern has little impact on the average hard brake ratio and throughput ratio. Regarding CPI, implementing this pattern leads to a value 0.88405, very close to that of double-step plaza with only basic merging patterns. Therefore, we conclude that the basic move-towards-travel-lane pattern is enough to ensure a satisfactory performance.

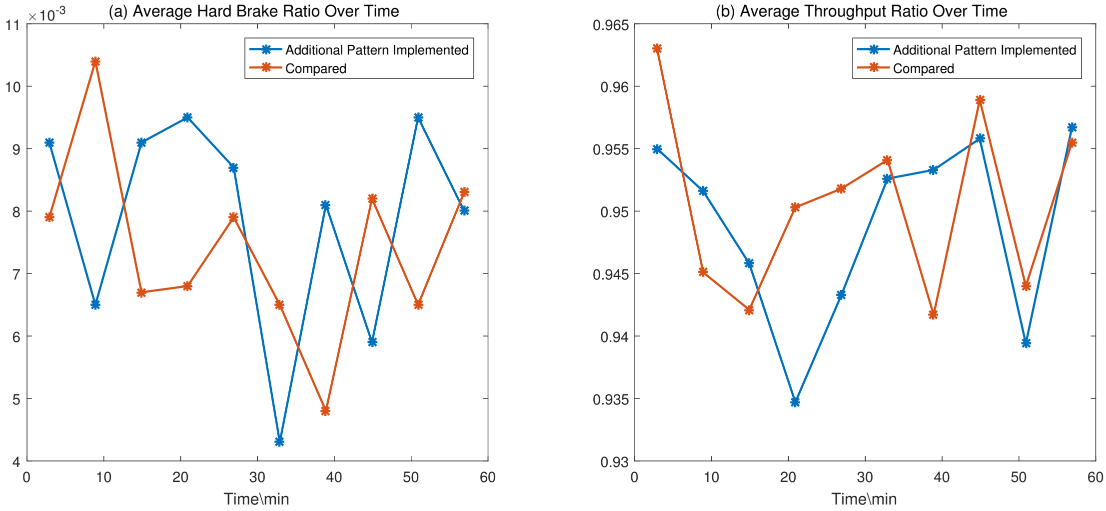


Figure 7: Comparison between additional merging pattern implementation. (a) Average hard braking ratio and (b) average throughput ratio over time. Ten samples points marked by asterisk denote average values per 10 mins.

3.3 Traffic Conditions

We then evaluate the performance of our double-step design in various traffic conditions. The normal traffic condition, as the default traffic condition for simulation, is 2800 cars per hour. We compare its results with light (2000 cars per hour) and heave (3600 cars per hour) cases. The results are shown in Figure 8.

Traffic conditions has little influence on hard brake ratios. For throughput ratio, light traffic and normal traffic have similar results (0.94584 vs 0.95065), while heave traffic leads to generally lower throughput (0.92926). The reason may be that for both light and normal traffic, cars don't queue up before a booth.

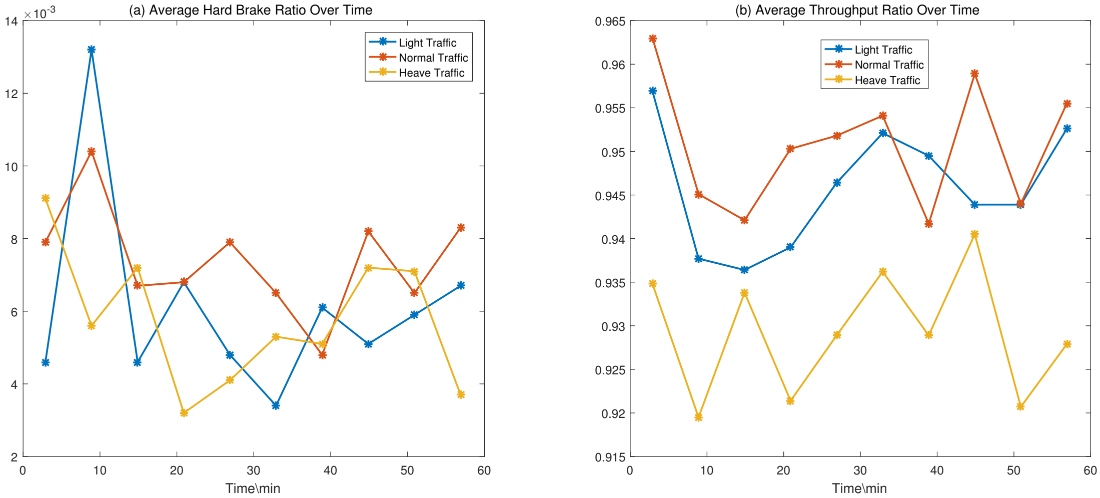


Figure 8: Comparison between different traffic conditions. (a) Average hard braking ratio and (b) average throughput ratio over time. Ten samples points marked by asterisk denote average values per 10 mins.

CPI for heave and light traffic are 0.9015 and 0.88528, indicating our double-step design is insensitive to traffic conditions.

3.4 Proportions of Booths

The default ratio of booths, $E : A : M = 1 : 2 : 1$ is suggested by [9] for safety and throughput reasons. We consider two different proportions, $E : A : M = 2 : 1 : 1$ and $E : A : M = 1 : 1 : 2$, and analyze its effect on our design. The results are shown in Figure 9.

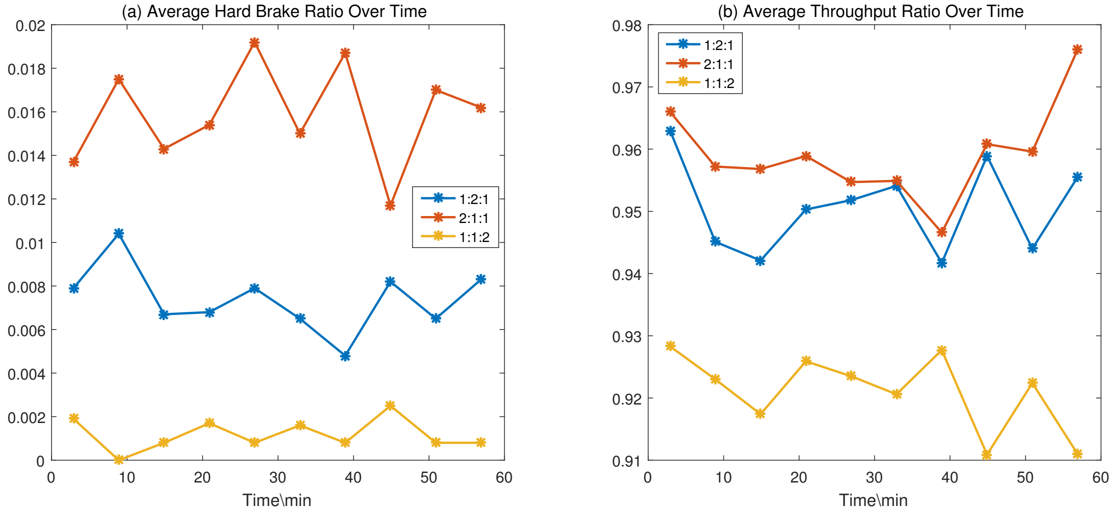


Figure 9: Comparison between different proportions of booths. (a) Average hard braking ratio and (b) average throughput ratio over time. Ten samples points marked by asterisk denote average values per 10 mins.

Large proportion of electronic booths leads to higher average speed, along with more hard brakes, as cars leaving from electronic booths have high initial speed and may be forced to take immediate stop due to low-speed vehicles from other lanes blocking the way. More human-staffed booth makes hard brakes much less frequent, but also generally lower speed. It is noteworthy, however, that although the average speed of a $1:2:1$ ratio is slightly lower than that of a $2:1:1$ (0.95065 vs 0.95915), the probability of accidents decreased dramatically (from 0.01587 to 0.0074). This is consistent with the suggestion from [9].

CPI for $2:1:1$ and $1:1:2$ are 0.88172 and 0.92107 respectively, which are still quite close to 0.88175, indicating our optimal design is not sensitive to proportions of booths.

3.5 Autonomous Vehicles

Compared to human drivers with a constant reaction time and limits in estimating distance and speed, autonomous vehicles can act without any 'hesitation' and detect surroundings with keen accuracy, thanks to its on-board computer system and advanced sensory techniques [10]. In our model, a safe following distance $\lfloor v_c \rfloor \Delta t$, is defined in speed update rule in Subsection 2.3. For autonomous cars, this distance is basically unnecessary. Therefore, by modifying our speed update rule, autonomous cars can be easily incorporated into our model.

3.6 Practical Implementation

As illustrated in previous subsections, the double-step design exhibits superior performance to others. The real-life implementation of such a design is double-slant design, as shown in Figure 10. The size of the plaza is approximately that of the double-slant plaza, 68 cells, or 1450.1 square yards.

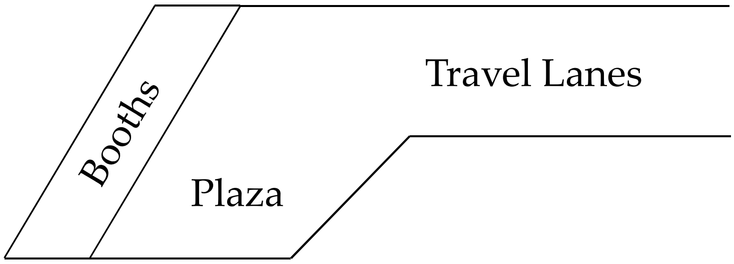


Figure 10: Graphic illustration of double-slant plaza, the real-life implementation of double-step design.

4 Final Remarks

4.1 Strengths and Weaknesses

We develop a cellular-automation model to simulate the traffic of merging area in a toll plaza and formulate a composite index, CPI to evaluate its performance. This leads to a set of strengths and weaknesses.

Strengths:

- **Applies widely.** Our model is designed to be applicable to a variety of scenarios. Different shapes and merging patterns of toll plazz, various traffic conditions and proportions of different tollbooths can be easily considered.
- **Captures key features of actual situations.** Drivers' strategy of lane-changing in fan-in area is well incorporated into our model. Characteristics of different types of booths are faithfully modeled.
- **A general evaluation metric.** We use three dimensionless scenario-independent metrics to formulate CPI. This allows us to circumvent the problem of inaccuracy in parameters and compare designs of different toll plaza schemes.

Weaknesses:

- **Relies on simulation.** Compared to macroscopic approaches of traffic flow analysis, our model relies on simulation results and can't reach closed-form solutions.
- **Computationally expensive.** The aggregate traffic behaviors, such as throughput, are computed from simulated traffic dynamics, which require intensive computational efforts.

4.2 Future Development

In the future, we hope to further develop and optimize our model in the following ways:

- **Rework the deceleration process.** In practice, deceleration has its limit and reaching beyond that leads to accidents. This limit should be incorporated into our model. In addition, random deceleration can be introduced to better illustrate characteristics of traffic flow.
- **Develop more realistic strategies for drivers.** Drivers in our model make decisions based on information of last time slot solely. Implementing look-ahead behaviors can be beneficial.
- **Consider the fan-out area.** The focus of our model is the design of the merging area of a toll plaza. By incorporating the fan-out area in front of booths, an overall optimal design for a toll plaza can be found.

4.3 Conclusion

In this report, we model the optimal design of merging area in a toll plaza. Firstly, we develop a cellular-automation model to simulate the traffic flow. Cars leaving rule, speed update rule, lane-changing rule and position update rule are introduced to adapt the basic cellular-automation model to our specific condition. A general evaluation metric, Composite Performance Index (CPI) is then formulated as a weighted composition of three dimensionless scenario-independent ratios: land utilization ratio, hard brake ratio and throughput ratio. Next, we implement several shapes of plaza and consider additional merging patterns. Sensitivity analysis is conducted with respect to traffic conditions, proportion of booths and introduction of autonomous vehicles. Ultimately, our model demonstrates a purposely designed shape, a 1450.1-square-yard double-slant plaza, along with the basic merging pattern introduced in lane-changing rule, outperforms other widely-used conditions.

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Future Technology*:
Independent Mathematical Consulting Services

Future Technology, L.L.C.*
Letter to New Jersey Turnpike Authority
1/23/17

To whom it may concern:

Traffic congestion has been a headache ever since the invention of vehicles. Modern transportation infrastructure, such as overpasses and flyovers, may relieve the problem to some extent but never solve it. A general approach to improve transportation capacity is to locate the bottleneck and redesign it, so that it can work in a more efficient manner. To this end, we develop a potent model to analyze the traffic flow of the merging area of a toll plaza, the most infamous contributor of a highway congestion. Through it, insight of a seemingly 'strange' yet highly operative toll plaza pattern has been found.

New Jersey may have the most complicated traffic conditions in the US. Considering adding additional lanes on major roads would be logistically impossible, we find it necessary to present our model to you, the management of Garden State Parkway and New Jersey Turnpike, in hope to facilitate traffic of toll plazas along them. As our model involves minimum scenario-specific factors, it can cater nicely to all barrier tolls in your highways.

Our model begins with the simulation of traffic flows in the toll plaza area. Once the basic properties of a toll plaza, such as number of egress lanes and travel lanes, types of booths, are given, reliable traffic data can be generated for different plaza designs.

To measure the overall performance of a possible design, we formulate a general performance index, which faithfully incorporates several key factors of toll plaza, including cost, throughput and accident prevention capability.

We then implement several widely-used plaza shapes, including the rectangular plaza and single-slant plaza. Unfortunately, their performance is unsatisfactory. Based on our analysis, a double-slant plaza, as shown below, is purposely designed. As our expected, this design exhibits extrodinary performance compared to ordinary ones, and is thus suggested by us as an alternative design for current tolls.

We hope our findings may be of help in dealing with troublesome traffic congestions in highway tolls.

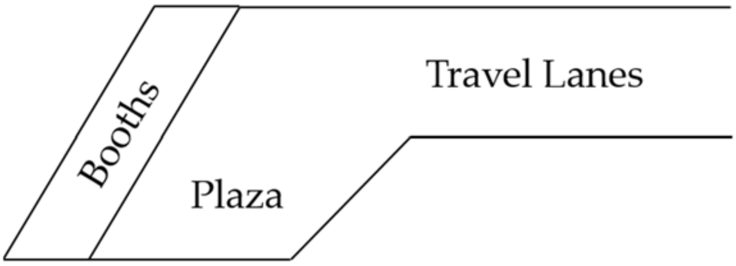

Figure: Double-slant Plaza Design

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*This company is purely a work of fiction.